Introduction

Example

Compute:

(a) $(1+3\sqrt{2})+(-3+2\sqrt{2})$

(b) $(2+2\sqrt{2})-(4-\sqrt{2})$

(c) $(3+\sqrt{2})(4-2\sqrt{2})$

(d) $\dfrac{1+\sqrt{2}}{-3+2\sqrt{2}}$

(a) $(1+3\sqrt{2})+(-3+2\sqrt{2}) = -2+5\sqrt{2}$

(b) $(2+2\sqrt{2})-(4-\sqrt{2}) = -2+3\sqrt{2}$

(c) $(3+\sqrt{2})(4-2\sqrt{2}) = 12-6\sqrt{2}+4\sqrt{2}-4 = 8-2\sqrt{2}$

(d) $\dfrac{1+\sqrt{2}}{-3+2\sqrt{2}} = \dfrac{(1+\sqrt{2})(-3-2\sqrt{2})}{(-3+2\sqrt{2})(-3-2\sqrt{2})} = \dfrac{-3-2\sqrt{2}-3\sqrt{2}-4}{9-8} = -7-5\sqrt{2}$

Example

Suppose $x^2 = 2$, compute

(a) $(1+3x)+(-3+2x)$

(b) $(2+2x)-(4-x)$

(c) $(3+x)(4-2x)$

(d) $\dfrac{1+x}{-3+2x}$

We introduce a new number, i, such that $i^2 = -1$, and write numbers as $a + ib$ where $a, b \in \mathbb{R}$.

Fact — All the normal rules of addition and multiplication still work

$$(1+i) + (4+5i) = (1+4) + (1+5)i = 5 + 6i$$

$$\begin{aligned}(3+2i)\times(4+5i) &= 3\times(4+5i) + 2i\times(4+5i)\\ &= 12 + 15i + 8i - 10\\ &= 2 + 23i\end{aligned}$$

Example

Solve the equation $(2+i)z = 3+4i$

$$\begin{aligned}z &= \frac{3+4i}{2+i}\\ &= \frac{3+4i}{2+i}\frac{2-i}{2-i}\\ &= \frac{(3+4i)(2-i)}{2^2+1^2}\\ &= \frac{10+5i}{5}\\ &= 2+i\end{aligned}$$

Definition. If $z = a + ib \in \mathbb{C}, a, b \in \mathbb{R}$ then we call a the **real part** of z, $a = \text{Re}(z)$ and b the **imaginary part** of z, $b = \text{Im}(z)$.

Definition. If $z = a + ib \in \mathbb{C}, a, b \in \mathbb{R}$ is a complex number, then its complex conjugate $z^* = a - ib$

Example

Prove that:

(a) $(z+w)^* = z^* + w^*$

(b) $(zw)^* = z^* w^*$

(c) $(z^*)^* = z$

(d) $a^* = a$ if $a \in \mathbb{R}$

Suppose $z = a + ib$ and $w = c + id$, then

(a)

$$\begin{aligned}(z+w)^* &= (a+c+i(b+d))^* \\ &= a+c-i(b+d) \\ &= a-ib+c-id \\ &= z^*+w^*\end{aligned}$$

(b)

$$\begin{aligned}(zw)^* &= ((ac-bd)+i(bc+ad)) \\ &= (ac-bd)-i(bc+ad) \\ z^*w^* &= (a-bi)(c-di) \\ &= (ac-bd)-i(bc+ad)\end{aligned}$$

Example

Find the roots of $z^2 + 4z + 5 = 0$

$$
\begin{aligned}
0 &= z^2 + 4z + 5 \\
&= (z+2)^2 + 1 \\
\Rightarrow \quad -1 &= (z+2)^2 \\
\Rightarrow \quad \pm i &= z + 2 \\
\Rightarrow \quad z &= -2 \pm i
\end{aligned}
$$

Example

If f is a polynomial with *real* coefficients then it's roots come in complex-conjugate pairs

Suppose $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$, *then*

$$\begin{aligned} 0 &= f(z) \\ 0^* = 0 &= f(z)^* \\ &= (a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0)^* \\ &= a_n{}^* (z^n)^* + a_{n-1}{}^* (z^{n-1})^* + \cdots + a_0{}^* \\ &= a_n z^{n*} + a_{n-1} (z^{n-1})^* + \cdots + a_0 \\ &= a_n (z^*)^n + a_{n-1} (z^*)^{n-1} + \cdots + a_0 \\ &= f(z^*) \end{aligned}$$

Therefore if $z \in \mathbb{C}$ *is a root,* z^* *is also a root.*

Example

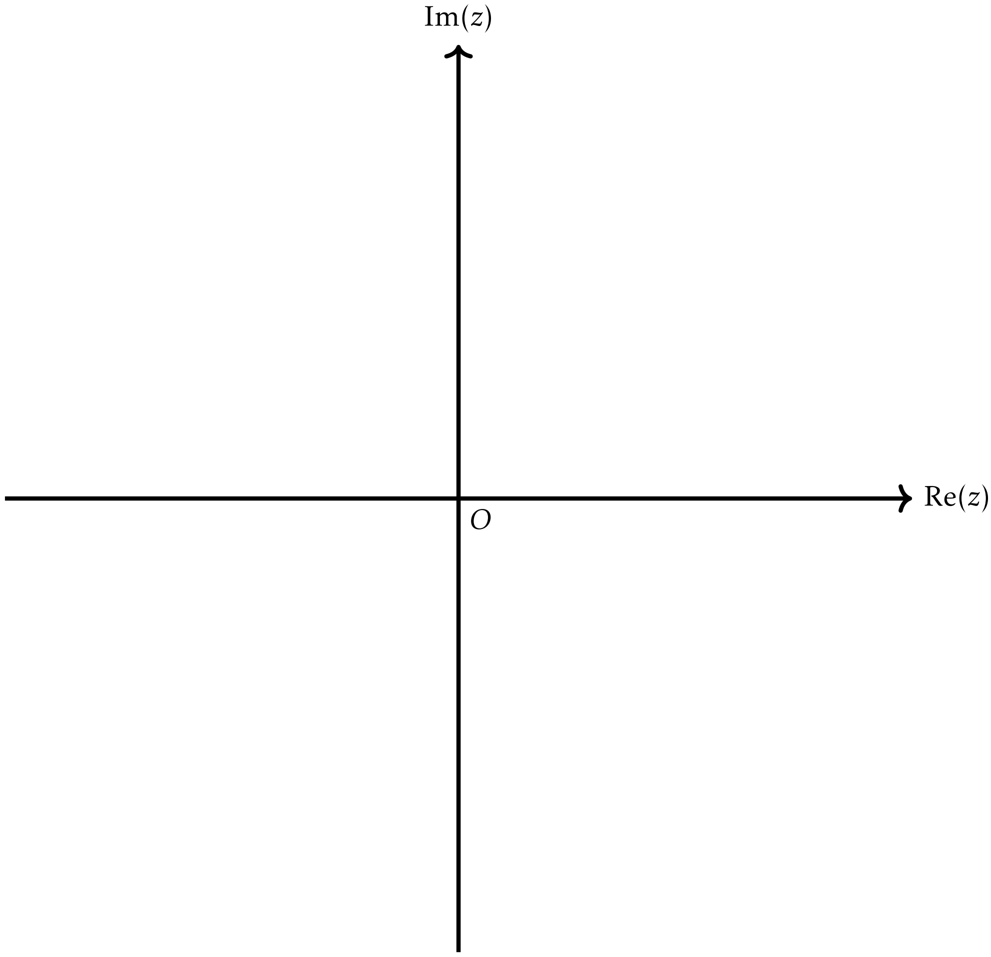
Solve $z^2 + i = 0$

Let $z = a + ib$, *then we have*

$$\begin{aligned} && 0 &= a^2 - b^2 + 2ab\,i + i \\ \Rightarrow && 0 &= a^2 - b^2 \\ && -1 &= 2ab \\ \Rightarrow && 0 &= a^2 - \left(\frac{-1}{2a}\right)^2 \\ && &= a^2 - \frac{1}{4a^2} \\ \Rightarrow && 0 &= 4a^4 - 1 \\ \Rightarrow && a &= \pm\frac{1}{\sqrt{2}} \\ && b &= \mp\frac{1}{\sqrt{2}} \\ \Rightarrow && z &= \pm\frac{1-i}{\sqrt{2}} \end{aligned}$$

Fact (Algebraic Closure) — If $f \in \mathbb{C}[X]$ is a polynomial with coefficients in $\mathbb{C}$ then it has a complex root.

The Complex Plane



Definition. The **modulus** of a complex number, $z = a + bi \in \mathbb{C}, a, b \in \mathbb{R}$,

$$|z| = \sqrt{a^2 + b^2}$$

Definition. The **argument** of a complex number, $z = a + bi \in \mathbb{C}, a, b \in \mathbb{R}$, is the angle in *radians* measured anticlockwise from the positive real axis.

$$\arg(z) = \arctan\left(\frac{b}{a}\right) \tag{*}$$

$\arg(z) \in (-\pi, \pi]$ or $\arg(z) \in [0, 2\pi)$.

Example
Find the modulus and argument of $1 + i$, $2i$, $-1 + 2i$, $-1 - i$.

- $1 + i$: $|1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2}$, $\arg(1 + i) = \arctan(1/1) = \frac{\pi}{4}$
- $2i$: $|2i| = 2$, $\arg(2i) = \frac{\pi}{2}$
- $-1 + 2i$: $|-1 + 2i| = \sqrt{1 + 4} = \sqrt{5}$, $\arg(-1 + 2i) = \pi - \arctan(2) \approx 2.03$
- $-1 - i$: $|-1 - i| = \sqrt{2}$, $\arg(-1 - i) = -\frac{3\pi}{4}$ *(or $\frac{5\pi}{4}$)*

Fact — If $z_1, z_2 \in \mathbb{C}$ then $|z_1 z_2| = |z_1||z_2|$ and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \pmod{2\pi}$

Example
Find the complex number with modulus 2 and argument $\frac{\pi}{3}$

Definition. The **modulus-argument form** of a complex number is

$$z = r(\cos\theta + i\sin\theta) = r\text{cis}\theta$$

Example

Show that $\sin(A+B) = \sin A\cos B + \cos A\sin B$ and find a similar formula for $\cos(A+B)$

Complex Geometry

Example

Show that $|z+w| \le |z| + |w|$

Geometrically, this is the triangle inequality: the length of one side of a triangle is at most the sum of the other two sides.

Algebraically: Let $z = a+bi$, $w = c+di$. Then

$$\begin{aligned}|z+w|^2 &= (a+c)^2 + (b+d)^2 \\ &= a^2 + 2ac + c^2 + b^2 + 2bd + d^2 \\ &= |z|^2 + |w|^2 + 2(ac+bd) \\ &\le |z|^2 + |w|^2 + 2\sqrt{(a^2+b^2)(c^2+d^2)} \quad \textit{(Cauchy-Schwarz)} \\ &= |z|^2 + |w|^2 + 2|z||w| = (|z|+|w|)^2\end{aligned}$$

Taking square roots: $|z+w| \le |z| + |w|$.

Example

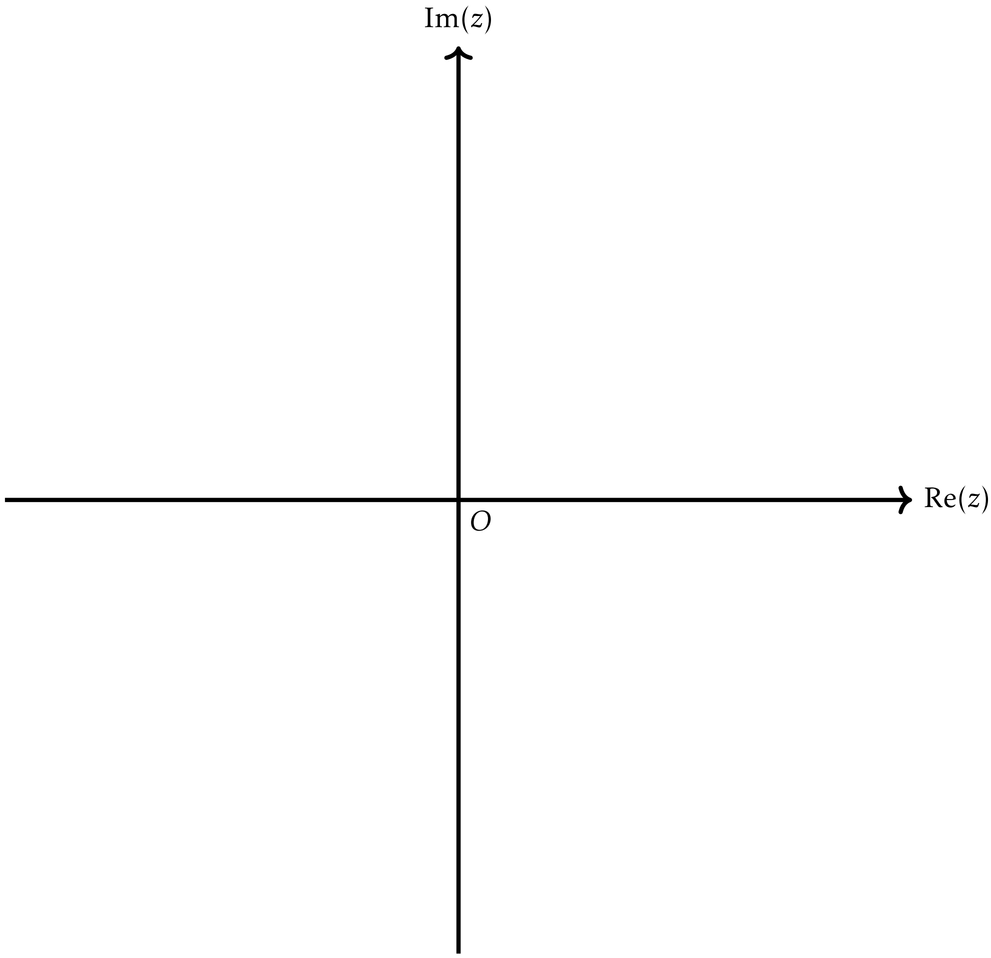
Sketch

(a) $|z-(1+i)| = 2$

(b) $\arg(z-1) = \frac{\pi}{3}$

(c) $\mathrm{Re}(z) = 3$

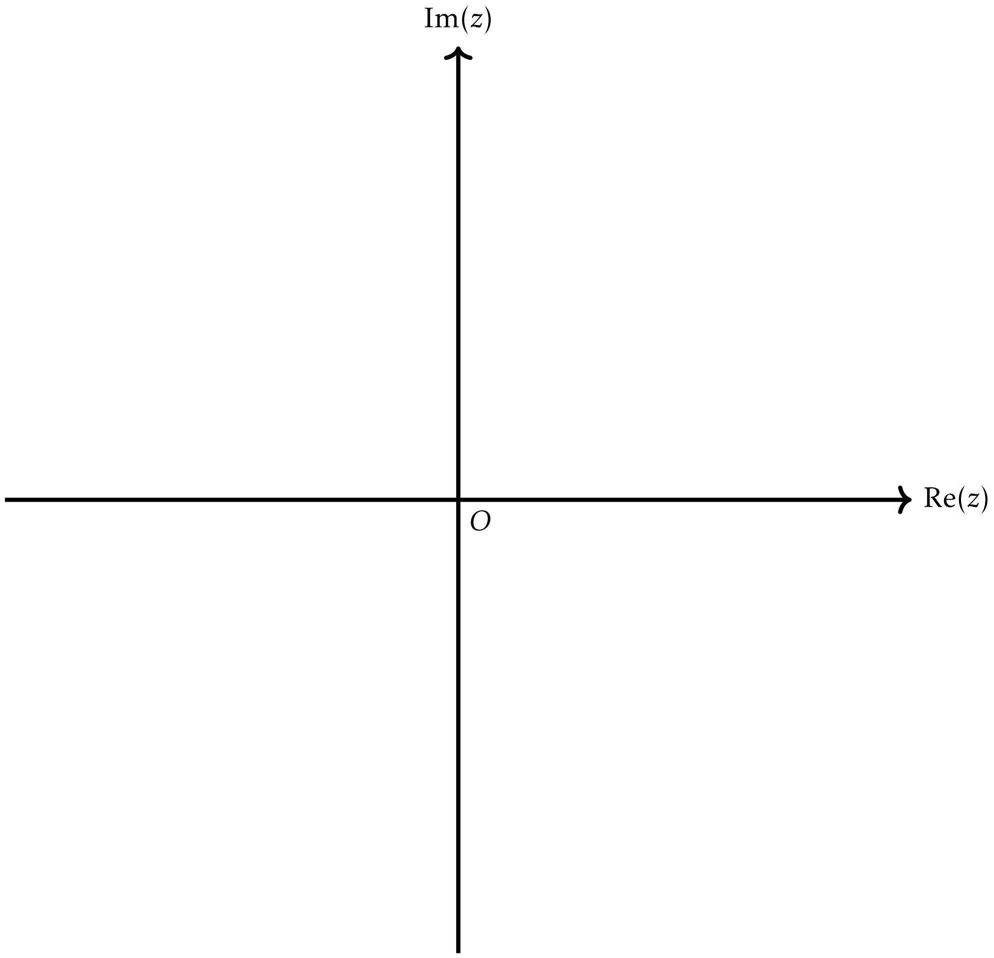
(d) $|z-1| = |z-i|$


Im(z)
Re(z)
O

Example

Sketch

(a) $|z-1| = 2|z-i|$



Actual Complex Geometry

Fact (Pro Tips) —

- Addition is translation
- Multiplication by a positive real number is enlargement
- Conjugation is reflection in the real-axis.
- Multiplication by cisθ is (anti-clockwise) rotation about the origin by θ

Example (Bottema's Theorem)
in any given triangle ABC, construct squares on any two adjacent sides, for example AC and BC. The midpoint of the line segment that connects the vertices of the squares opposite the common vertex, C, of the two sides of the triangle is independent of the location of C

Example (Napoleon's Theorem)

On the exterior of triangle ABC three new equilateral triangles $AC'B$, $BA'C$ and $CB'A$ are constructed. Prove that the centroids of these triangles are the vertices of an equilateral triangle